# Lance CH 4 (Tensor Products)

Tensor prod of vector spaces:

$V, W$ vector spaces (over $\mathbb{C}$)

→
$$\begin{tikzcd} V \otimes W \arrow[r, "L\ \text{linear}"] & Z \\ V \times W \arrow[u, dotted, "T"] \arrow[ur, "B\ \text{bilinear}"'] & \end{tikzcd}$$

Construct $V \otimes W$ as the span of $\{v \otimes w \mid v \in V, w \in W\}$ mod the subspace spanned by

$$\left\{ (av_1 + bv_2) \otimes w - \big(a(v_1 \otimes w) + b(v_2 \otimes w)\big), \quad v \otimes (aw_1 + bw_2) - \big(a(v + w_1) + b(v \otimes w_2)\big) \right\}$$

- e.g. $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$
- This is called the "algebraic" tensor product : $V \otimes_{alg} W$

## Tensor prod of Hilbert Spaces:

- $H, K$ Hilbert spaces
- Define inner product on $H \otimes_{alg} K$:

  $$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{H \otimes K} = \langle v_1, v_2 \rangle_H \langle w_1, w_2 \rangle_K$$

- $H \otimes_{alg} K$ not complete w.r.t. norm induced by $\langle -, - \rangle_{H \otimes K}$
- Define $H \otimes K$ to be the completion of $H \otimes_{alg} K$ w.r.t. norm from inner prod.

## Tensor prod of $C^*$-algebras:

- $A, B$ $C^*$-algebras
- Define $*$-algebra structure on $A \otimes_{alg} B$

  $$(a \otimes b)(c \otimes d) = ac \otimes bd$$

  $$(a \otimes b)^* = a^* \otimes b^*$$

Goal: embed $A \otimes_{alg} B$ as a $*$-subalgebra of $B(H)$ for some $H$, use norm on $B(H)$.

Lemma: $B(H) \otimes B(K) \hookrightarrow B(H \otimes K)$

$$S \otimes T(h \otimes k) = Sh \otimes Tk \qquad \begin{array}{l} S \in B(H) \\ T \in B(K) \\ h \otimes k \in H \otimes K \end{array}$$

Let $\pi : A \to B(H)$, $\eta : B \to B(K)$ be faithful non-degenerate representation

Get $*$-hom:

$$\begin{array}{ccc} A \otimes_{alg} B & \hookrightarrow & B(H \otimes K) \\ & & \rotatebox{90}{$\in$} \\ a \otimes b & \longmapsto & \pi(a) \otimes \eta(b) \end{array}$$

Define norm on $A \otimes_{alg} B$ by

$$\|t\|_\sigma = \|(\pi \otimes \eta)(t)\|_{B(H \otimes K)}$$

Define $A \otimes_\sigma B$ to be completion of $A \otimes_{alg} B$ w.r.t. $\|-\|_\sigma$.

"spatial tensor product"

WARNING!!!

There can exist other $C^*$-norms on $A \otimes_{alg} B$ which give non-isomorphic completions. e.g.

$$\|t\|_{max} := \sup \left\{ \|t\|_\gamma \;\middle|\; \begin{array}{l} \|-\|_\gamma \text{ is a } C^*\text{-norm} \\ \text{on } A \otimes_{alg} B \end{array} \right\}$$

- Nuclear $C^*$-algebra

examples:

- $M_n(\mathbb{C}) \otimes A \cong M_n(A)$

  $(\lambda_{ij}) \otimes a \longmapsto (\lambda_{ij} a)$

- $C_0(T) \otimes A \cong C_0(T, A)$

  $f \otimes a \longmapsto (t \mapsto f(t)a)$

- $A \otimes K(H)$

# Exterior Tensor Prod of Hilbert $C^*$-modules

- $A, B$ $C^*$-algebras
  $E$ a Hilbert $A$-module
  $F$ a Hilbert $B$-module

Start with $E \otimes_{alg} F$ ...

This is a right $(A \otimes_{alg} B)$-module

$$(x \otimes y)(a \otimes b) = xa \otimes yb$$

$$x \in E,\ y \in F \qquad a \in A,\ b \in B$$

Inner product: $(A \otimes_{alg} B)$-valued

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \underbrace{\langle x_1, x_2 \rangle_E}_{A} \otimes \underbrace{\langle y_1, y_2 \rangle_F}_{B}$$

Need to complete: (CH 1)

- Complete $A \otimes_{alg} B$ to $A \otimes B$ (spatial tensor prod)
- Norm on $E \otimes_{alg} F$ comes from inner prod above.
- $A \otimes_{alg} B$-module structure extends to $A \otimes B$ by continuity: $\|xa\|_{E \otimes F} \leq \|x\|_{E \otimes F} \|a\|_{A \otimes B}$
- $E \otimes F$ is completion, a Hilbert $A \otimes B$-module.

e.g. $\mathbb{C} \otimes A \cong A$

If $H$ is a Hilbert space $H_A \cong H \otimes A_A$

$H$: $\mathbb{C}$-module, $A_A$: $A$-module

Similar to Hilbert spaces get embedding:

$$j : L(E) \otimes L(F) \hookrightarrow L(E \otimes F)$$

exterior

If $A = B = \mathbb{C}$, this is same embedding.

$$j(K(E) \otimes K(F)) = K(E \otimes F)$$

$$\Theta_{u,v} \otimes \Theta_{x,y} = \Theta_{u \otimes x,\, v \otimes y} \quad \text{(exercise)}$$

If $E = A_A$, $F = B_B$, then this is an embedding:

$$j : M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$$

# Interior Tensor prod of Hilbert $C^*$-modules:

Same setup as exterior tensor prod

also given a $*$-hom

$$\phi : A \to L(F)$$

We'll construct $E \otimes_\phi F$ which will be a Hilbert $B$-module (not $(A \otimes B)$-module)

We can define left $A$-module structure on $F$ by:

$$a \cdot y \longmapsto \phi(a) y \qquad a \in A,\ y \in F$$

Start with:

$$E \otimes_A F = E \otimes_{alg} F \Big/ \operatorname{span}\{ xa \otimes y - x \otimes \phi(a) y \}$$

this is a right $B$-module:

$$(x \otimes y) b = x \otimes (yb)$$

have $B$-valued inner prod:

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \phi(\langle x_1, x_2 \rangle_E) y_2 \rangle_F$$

Check that:

If $z = xa \otimes y - x \otimes \phi(a) y$ then

$$\langle z, z \rangle = 0$$

$E \otimes_\phi F$ is completion of $E \otimes_A F$ w.r.t. norm induced by inner prod.

e.g. $\phi \in Mor(A, K(F))$ $\quad$ $F$ is a Hilbert $B$-module

$\phi : A \to M(K(F)) \cong L(F)$ $\quad$ $\underline{\phi(A)F \text{ dense in } F}$

what is $H_A \otimes_\phi F$ ?

$$\underset{\substack{\uparrow \\ H_A}}{(\xi \otimes a)} \otimes y \longmapsto \xi \otimes \phi(a) y$$

$$H_A \otimes_\phi F \cong \underline{\underline{H \otimes F}}$$